Text Chapters 3 & 4

Steady Waves

Speed of Sound

Speed of Sound, \( a \) \hspace{1cm} (3.1)

Speed of Sound for a Perfect Gas \hspace{1cm} (3.2)

The speed of sound, \( a \), is the speed at which disturbances propagate through a fluid.

**Disturbances from \( a \)**

The distance traveled by a disturbance is the speed of sound times the time since being emitted. The waves in the diagram were emitted at 1t time increments. If they come from the same location the disturbance fronts will form concentric circles. This occurs when you drop something in water for example.

**Disturbance from \( a \)**

Again, at, 2at, 3at represent distances traveled by the emitted disturbance and the disturbance fronts fall within one another. This is the situation when the flow is subsonic.
Disturbance from a

- The region which has felt the disturbances emitted in the past.
- The region which is not influenced by current or past disturbances.
- The line/surface that divides the two zones.

\[
\sin \mu = \frac{at}{Vt} = \frac{1}{M}
\]

\[
\mu = \sin^{-1} \left( \frac{1}{M} \right) \quad (3.3)
\]

Disturbances from a
Derivation of Speed of Sound

The derive the speed of sound from the governing equations assume a wave of infinitesimal strength.

\[ \rho a = (\rho + dp)(a + da) \]

\[ a = -\rho \frac{da}{dp} \quad (3.4) \]

\[ p + \rho a^2 = (p + dp) + (\rho + dp)(a + da)^2 \]

\[ 0 = dp + a^2 dp - 2 \rho a da \quad (3.5) \]

\[ 0 = dp + a^2 dp - 2 a^2 dp \]

\[ a^2 = \frac{dp}{dp} = \left( \frac{dp}{dp} \right)_s \]
Waves

Shock waves form when the infinitesimal sound waves coalesce into a finite wave.

**Normal Shock Waves**

Property Changes from 1 to 2

\[ \uparrow \text{ (go up):} \]

\[ \downarrow \text{ (go down):} \]

Stay the same:

Again properties are related by applying the governing equations to both sides. The result can be written as functions of Mach number.

\[ M_2^2 = \frac{1 + \frac{\gamma - 1}{2} M_1^2}{\gamma M_1^2 - \frac{\gamma - 1}{2}} \quad (3.6) \]

\[ \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2} \quad (3.7) \]

\[ \frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1) \quad (3.8) \]
Note that equations (3.6) to (3.11) are tabulated in the text.

* State

A sometimes useful condition is known as the * state. It is . This is similar to the stagnation state where properties are brought to rest isentropically.

\[ T^* = \text{Temperature obtained if the flow is} \]

\[ a^* = \text{Speed of sound obtained from} \]
Derivation of Normal Shock Equations

The one dimensional flow governing equations can be written

\[ \rho_1 u_1 = \rho_2 u_2 \]
\[ p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \]
\[ h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} = h_o \]

The energy equation \( h_o = \text{constant} \) can be rewritten for a calorically perfect gas using the relations

Then

\[ \frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2} \]

\[ \frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = a_1^* + \frac{a_2^*}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a_2^* \]  \hspace{1cm} (3.12)

\[ \left( \frac{a_1}{u_1} \right)^2 + \frac{(\gamma - 1)}{2} = \frac{\gamma + 1}{2} \left( \frac{a_1^*}{u_1} \right)^2 \]  \hspace{1cm} (3.13)

\[ \frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1 \]

\[ \frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1 \]
\[ \frac{\gamma+1}{2\gamma u_1 u_2} a^* u_1 - \frac{\gamma-1}{2\gamma} u_1^2 - \frac{\gamma+1}{2\gamma u_2} a^* u_2 + \frac{\gamma-1}{2\gamma} u_2^2 = u_2 - u_1 \]

\[ \frac{\gamma+1}{2\gamma u_1 u_2} (u_2 - u_1) a^* + \frac{\gamma-1}{2\gamma} (u_2 - u_1) = u_2 - u \]

\[ \frac{\gamma+1}{2\gamma u_1 u_2} a^* + \frac{\gamma-1}{2\gamma} = 1 \]

\[ a^* = u_1 u_2 \quad (3.14) \]

\[ 1 = \frac{u_1 u_2}{a^* a^*} = M_1^* M_2^* \]

\[ M_2^* = \frac{1}{M_1^*} \quad (3.15) \]

\[ \frac{\gamma+1}{2 + (\gamma-1)M_2^2} = \frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2} \]

\[ \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_1 u_2} = \frac{u_1^2}{a^*} = M_1^{*2} \]

\[ p_2 - p_1 = \rho_1 u_1 (u_1 - u_2) = \rho_1 u_1^2 \left( 1 - \frac{u_2}{u_1} \right) \]

\[ \frac{p_2 - p_1}{p_1} = \frac{\rho_1 u_1^2 \left( 1 - \frac{u_2}{u_1} \right)}{p_1} = \gamma M_1^2 \left( 1 - \frac{u_2}{u_1} \right) \]

\[ \frac{T_2}{T_1} = \left( \frac{p_2}{p_1} \right) \left( \frac{\rho_1}{\rho_2} \right) \]
Oblique Shock Waves

When a flow is itself an oblique shock wave forms. This is a consequence of the inviscid wall tangency boundary condition.

One dimensional supersonic theory tells us that
\[ M_1 > M_2 \rightarrow \]

Therefore since the wall turns the fluid, disturbances caused by the wall can result in changes within the zone of silence for points directly ahead of the turn. In addition, since disturbances from both before and after the corner can affect flow in this region the shock forms between the two Mach lines. The equations which describe this flow are exactly the normal shock relations (with the exception of the total pressure) if the Mach number normal to the shock wave is used.

So far we have made no statements about \( \beta \). The most that can be said thus far is that it must be \( \mu_1 \) and \( \mu_2 + \theta \).

Consider then the above diagram and assume a straight oblique shock wave on either side of which exist constant properties. The velocity vectors can be resolved into components that are to the shock wave.
Now to a control volume that is aligned with the shock wave and falls between two streamlines.

Recall the steady continuity equation

\[
\nabla \cdot \mathbf{d} \mathbf{s} = \begin{bmatrix} A \\ D \\ B,C,E,F \end{bmatrix}
\]

(4.1)

We have

\[
\int_{S} \rho \nabla \cdot \mathbf{d} \mathbf{s} \, dW = - \int_{S_{t}} \mathbf{n} \cdot \mathbf{d} \mathbf{s}
\]

(4.3)

Simplify

or

(4.4)
Simplify

or

\begin{equation}
\int_{s} (\rho \vec{V} \cdot d\vec{s}) u = - \int_{s_n} \rho \hat{n} \cdot d\vec{S}
\end{equation}

\begin{equation}
- \int_{s} \rho \vec{V} \cdot d\vec{s} = \int_{s} (\rho \vec{V} \cdot d\vec{s}) \left( e + \frac{V^2}{2} \right)
\end{equation}

\begin{align*}
- (-p_1 u_1 + p_2 u_2) &= - \rho_1 u_1 \left( e + \frac{V_1^2}{2} \right) + \rho_2 u_2 \left( e_2 + \frac{V_2^2}{2} \right) \\
\left( h_1 + \frac{V_1^2}{2} \right) \rho_1 u_1 &= \left( h_2 + \frac{V_2^2}{2} \right) \rho_2 u_2
\end{align*}

Note that, therefore, we can determine properties on either side of an oblique shock by applying the with an appropriate Mach number. We need
For a calorically perfect gas

\[
\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_{n_1}^2}{(\gamma - 1)M_{n_1}^2 + }
\]

\[
\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_{n_1}^2 - 1)
\]

\[
M_{n_2}^2 = \frac{M_{n_1}^2 + \frac{2}{\gamma - 1}}{\frac{2\gamma}{\gamma - 1}M_{n_1}^2 - 1}
\]

\[
\frac{T_2}{T_1} = \frac{P_2}{P_1} \frac{\rho_1}{\rho_2}
\]

From geometry we have \( M_2 = \), \( \tan \beta = \) and \( \frac{\tan(\beta - \theta)}{\tan \beta} = \).

Hence

\[
\frac{\tan(\beta - \theta)}{\tan \beta} =
\]

or

\[
\frac{\tan(\beta - \theta)}{\tan \beta} =
\]

Finally, we can write

\[
\tan \theta =
\]

(4.9)
Shock Polar

The Shock Polar is a graphical representation of the shock system in (x,y) velocity space, i.e. . For example, consider the geometry discussed previously during the oblique shock equation derivation. If the pre- and post-shock velocities are plotted in the hodograph plane they look like

If all possible shock solutions for a given \( V_1 \) are plotted together we get

Plots of this type were useful before hand calculators were available. They are now used less frequently, but demonstrate some aspects of shock waves that are important. In particular, there are as many as two solutions for a particular angle, there is a maximum angle that a shock wave can attain for given freestream conditions, and there are some angles for which no solution exists.
Formation of Shock Waves

coaalesce into a shock wave. Consider the geometry

What sort of wave is formed by a surface that *turns away from itself*?

Surface tangency requires that the flow turn parallel to the surface, but tells us nothing about the turning process. For that information we must turn to the governing equations. Consider the two possibilities shown above. First a discontinuous wave

With the associated control volume

Clearly the same analysis holds as for the oblique shock relations and hence also the same governing equations. In addition, the sketches show that $u_2 \neq u_1$, hence

$$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} 1$$  \hspace{1cm} (4.10)

and since

$$\frac{p_2}{p_1} = 1 + \frac{\rho_1 u_1^2}{p_1} \left( 1 - \frac{\rho_1}{\rho_2} \right)$$

we get

$$\frac{p_2}{p_1} 1$$  \hspace{1cm} (4.11)

And by employing a small angle entropy jump formula

$$s_2 - s_1 \approx \frac{R(\gamma + 1)}{12 \gamma^2} \left( \frac{\Delta p_1}{p_1} \right)^3$$  \hspace{1cm} (4.12)

we see that entropy

The alternative is then to assume an exists which emanates from the corner and consists of nonintersecting Mach lines.
Prandtl-Meyer Expansions

An isentropic relationship then exists which relates the flow angle, $\nu$, to the Mach number, $M$. A reference value is then prescribed, i.e. $\nu(M = 1) = 0$. This function $\nu(M)$ is known as the Prandtl-Meyer function and is given by the relation

$$\mu(m) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \sqrt{\frac{\gamma - 1}{\gamma + 1} (M^2 - 1) - \tan^{-1} \sqrt{M^2 - 1}}$$

(4.13)

Graphically, we have

The Prandtl-Meyer function is tabulated in the appendix. Conditions on either side and within the expansion wave are related via isentropic conditions once $\nu$, and (hence) $M$ are known.

Derivation of the Prandtl-Meyer Function

Before beginning some nomenclature is needed.
To get the accumulated effect of all Mach waves in an expansion fan of finite $\theta$, we

\[ V + \frac{dV}{V} = \frac{\sin\left(\frac{\pi}{2} + \mu\right)}{\sin\left(\frac{\pi}{2} - \mu - d\theta\right)} \]

\[ \sin\left(\frac{\pi}{2} + \mu\right) = \sin\left(\frac{\pi}{2} - \mu\right) = \cos \mu \]

\[ \sin\left(\frac{\pi}{2} - \mu d\theta\right) = \cos(\mu + d\theta) = \cos \mu \cos d\theta - \sin \mu \sin d\theta \]

\[ 1 + \frac{dV}{V} = \frac{\cos \mu}{\cos \mu \cos d\theta - \sin \mu \sin d\theta} \]

\[ 1 + \frac{dV}{V} = \frac{1}{1 - \tan \mu d\theta} \]

\[ 1 + \frac{dV}{V} = 1 + d\theta \tan \mu + \ldots \]

\[ d\theta = \frac{dV}{V \tan \mu} \]

\[ d\theta = \frac{dV}{V} \sqrt{M^2 - 1} \quad (4.14) \]

To get the accumulated effect of all Mach waves in an expansion fan of finite $\theta$, we

\[ (4.15) \]

\[ \ln V = \ln M + \ln a \]

\[ \frac{dV}{V} = \frac{dM}{M} + \frac{da}{a} \]
The integral in (4.16) is defined by (4.13).

\[ \frac{d}{dt} = -\left(\frac{\gamma-1}{2}\right) M \left(1 + \frac{\gamma-1}{2} M^2 \right)^{-1} dM \]

\[ \frac{dV}{V} = \frac{1}{1 + \frac{\gamma-1}{2} M^2} \frac{dM}{M} \]

\[ \theta_2 = \int_{\theta_1}^{\theta_2} \frac{M_2}{M_1} \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma-1}{2} M^2} \frac{dM}{M} \]

\[ \Theta_2 = \int_{\theta_1}^{\Theta_2} \frac{M_2}{M_1} \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma-1}{2} M^2} \frac{dM}{M} \]

\[ v(M) = \int \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma-1}{2} M^2} \frac{dM}{M} \quad (4.16) \]
Wave Interactions

Regular Reflection From a Solid Boundary

Consider the geometry shown below

Some mechanism must exist to condition in region (?). The question is; does the flow turn into itself or away from itself?

To visualize the situation, assume that the incipient shock does not exist, instead the wall before point P is in the direction of the post shock velocity (the upturned lower wall). The flow then sees the upper wall as a wedge that will turn the flow into itself. Hence

Consider the velocities in the three regions. Since conditions in region 2 are known, conditions in region 3 follow if $\theta_2$ is known. Note the above is based on the fact that the geometry shows the flow turning into itself. If the flow turns away from itself in region 3 is needed to determine conditions in region 3.
The above discussion holds for shock waves, however, expansion wave reflections have similar properties. However, these calculations are more difficult because expansion waves have a finite size. Determination of properties in these regions typically are performed using a discrete approach such as the method of characteristics to be discussed later.

Problem Children

Suppose that $\theta_2 > \theta_{\text{max}}$ for a particular $M_2$? In this case occurs. This type of reflection is required to satisfy . Note that the shock stem OW is curved. If the shape is known properties in region ROW can be determined from the jump conditions and the method of characteristics to be discussed later.
Wave Families
Waves can be segregated using the term [diagram] which deflects a stream to the left relative to someone traveling with the stream. Similarly deflect the stream to the right.

Left and right running waves constitute opposite families of waves. This designation is used when describing how waves interact and produces several general patterns of interaction.

Intersection of Waves of Opposite Families
Consider supersonic flow in a convergent nozzle geometry in which a left running shock wave is generated by the lower corner and a right running shock wave is generated by the upper corner. These waves interact as diagramed below.

[Diagram of wave interaction]
A mechanism exists apart from flow tangency to turn the flow parallel beyond point E in regions 4 and 4'. However, the formation of a shock depends only on Mach number and deflection angle. This system requires additional conditions to exist. Consider the following

1. Since $\theta_2 \neq \theta_3$ we can assume changes in regions 1-4 and 1-4' are.
2. Since conditions in 4 and 4' are separates the two regions.
3. The pressure is constant across a slip line, this fixes its location.
4. Flow tangency applies to the slip line

\[ \Phi_4 = \Phi = \Phi \quad \text{or} \quad \frac{V_4}{|\vec{V}_4|} = \frac{\vec{V}}{|\vec{V}|} \]

3 and 4 allow us to solve iteratively for regions 4 and 4'. This is done as follows

1. Regions 2 and 3 are known.
2. Guess $\Phi_4$.
3. Calculate $p_4 = p_{4'}$.
4. Calculate $\theta_{4'}$.
5. Check $\Phi_{4'} = \theta_2 - \theta_{4'} = \Phi_4 = \theta_4 - \theta_3$.

Stop if $\Phi_{4'} = \Phi_4$.

Go to 2 if $\Phi_{4'} \neq \Phi_4$.

(Possible $\theta_4$ choice from $\Phi_4 = \Phi_{4'}$, since it points in the right direction.)

Note that the process is not completed since the reflected shocks will hit the walls. At some point this will become more difficult because a Mach stem will occur.

Note also that we have assumed that conditions are such that attached shock waves will occur from the first intersection.
Intersection of Shocks of the Same Family

Recall that Mach lines near a shock wave look as depicted

\[ \begin{align*}
\mu_1 & \quad \beta \\
\mu_2 & \quad \beta - \theta
\end{align*} \]

So that if we consider the interaction of waves from the same family as shown below we know that wave A must intersect wave B since \( \beta_2 > \mu_2 \). Therefore conditions in regions 2 and 3 are easily calculable. However, conditions in region 5 are not the same as in region 3 since the flow passes through a different strength shock structure. We again require so a slip line forms. Unfortunately, the system is still over determined, since uniquely determines the pressure in region 5. To satisfy \( p \) constant a . It can be either and must be found via iteration.

1. Regions 1, 2 and 3 are known.
2. Guess \( \theta_5 = \theta_4 \).
3. Calculate \( p_5 \).
4. Determine if wave 3-4 is a shock or an expansion.
5. Calculate \( p_4 \).
6. Is \( p_4 = p_5 \)?

   Yes - Stop
   No - Return to 2.

Note: This assumes all shocks are attached.