Text Chapter 3

Hydrostatics

a) General Relations

A static fluid is one that has zero acceleration (i.e., stationary). This class of fluid mechanics is referred to as ________________. The fact that the fluid is stationary implies that only normal stresses can be present.

To analyze this sort of fluid we begin by considering Newton's 2nd law applied to a control volume.

However, since the fluid is stationary it is clearly not accelerating. Therefore, Newton's second law states

Recall from chapter 1 that the forces acting on a fluid are:

1. 
2. 
3. 

Since the fluid is stationary we have no shear forces. Why?

Therefore, the pressure forces must negate the gravitation force. The resulting equation for the distribution of pressure in a static fluid, where gravitation acts in the minus y direction, is

\[ p = -\rho g y + p_{ref} \quad (3.1) \]

The pressure is then said to be ________________ and is the _____ ___________. The pressure increases linearly with depth.
EXAMPLE PROBLEM 3.1

Consider a large body of water on a STD. What is the distribution of pressure with depth in the water? List the result in psia and psig.

The pressure equation is again equation (3.1)

$$ p = -\rho g y + P_{\text{ref}} $$

If we consider absolute pressure then $P_{\text{ref}} = P_{\text{STD}}$. We get

$$ p = -(1000 \text{ kg/m}^3) \cdot (9.8 \text{ m/s}^2) y + 1.01 \cdot 10^5 \text{ Pa} $$

$$ = -9800 y + 1.01 \cdot 10^5 \quad [y-m] $$
So that at 100 meters below the surface the pressure is approximately atmospheres.

If we consider gauge pressure $p_{\text{ref}} = 0$ and we get

\[
p = -\left(1.94 \text{slug/ft}^3\right) \cdot \left(32.2 \text{ft/sec}^2\right) y \\
= -\left(1.94 \text{slug/ft}^3\right) \cdot \frac{\text{lb}_f \cdot \text{sec}^2/\text{ft}}{\text{slug}} \cdot 32.2 \text{ft/sec}^2 \cdot y \\
= -62.5 y \left(\text{lb}_f/\text{ft}^2\right) \cdot \frac{1\text{ft}^2}{144 \text{in}^2} \\
= -0.434 y \left(\text{psig}\right) \left[y-\text{ft}\right] \\
= -0.0362 y \left(\text{psig}\right) \left[y-\text{in}\right]
\]

So that at 100 ft below the surface the pressure is approximately atmospheres.

QUESTION: How do we know that equation (3.1) is correct?
To prove that it is correct we start with an infinitesimal volume of size dx dy dz. Assume that you know the pressure and density at the center.

To get the pressure force acting on each surface, we need the pressure at the surface. We can get this from a Taylor series expansion. For example, to get the pressure at the \( x + \frac{dx}{2} \) and \( x - \frac{dx}{2} \) planes

\[
p(x + \frac{dx}{2}) = p(x) + \frac{dx}{2} \frac{\partial p}{\partial x} + \frac{dx^2}{4} \frac{\partial^2 p}{\partial x^2} + \ldots
\]

\[
p(x - \frac{dx}{2}) = p(x) - \frac{dx}{2} \frac{\partial p}{\partial x} + \frac{dx^2}{4} \frac{\partial^2 p}{\partial x^2} + \ldots
\]

Now since the pressure force is given by the equation

\[
F_{\text{pressure}} = p \cdot \text{Area}
\]

the pressure force in the x-direction is

\[
\Sigma F_x = p(x - \frac{dx}{2}) \cdot A(x - \frac{dx}{2}) + p(x + \frac{dx}{2}) \cdot A(x + \frac{dx}{2})
\]

\[
= (p(x) - \frac{\partial p}{\partial x} \frac{dx}{2} + \frac{\partial^2 p}{\partial x^2} \frac{dx^2}{8} + \ldots) A_x + (p(x) + \frac{\partial p}{\partial x} \frac{dx}{2} + \frac{\partial^2 p}{\partial x^2} \frac{dx^2}{8} + \ldots) A_x
\]

\[
= \Sigma F_y = \Sigma F_z =
\]
Similarly for the $y$ and $z$ directions, so that

$$\sum \vec{F}_{\text{pressure}} =$$

Next, gravitation forces are found from the vector $\vec{f}$, which is the gravity force per unit mass (N/kg), so that

$$\vec{F}_{\text{Gravity}} =$$

Therefore, from Newton's second law

and if the gravity force acts in the minus $y$ direction, we have

$$\rho (0 \hat{i} - g \hat{j} + 0 \hat{k}) = \frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k}$$

which leads to

$$(3.2a)$$
so that if \( \rho \) is a constant we get

\[
(3.2b)
\]

or

\[
p = -\rho gy + p_{ref}
\]

(3.1)

Then lines of constant pressure, isobars, are \( y = \text{const} \) lines, which are everywhere perpendicular to the applied gravitational force field.

This equation can also be used to determine the buoyancy forces on floating or submerged bodies.

Variable Density Fluids

In cases where the \( \rho = f(x,y) \) and is pressure independent, (i.e., salt water with salinity variations throughout), but resides in a gravitational field a similar equation is found

\[
p = \text{Const}_1 y + \text{Const}_2
\]

An example of this is the atmosphere in which both pressure and density vary due to the changes in temperature. For perfect gases pressure, density and temperature are linked through the perfect gas equation of state.
(Other gases relate these variables through different equations of state.) Where \( R \) is the gas constant for air

\[
R_{\text{air}} = 1716 \frac{\text{ft}^2}{\text{sec}^2 \circ R} = 286.9 \frac{\text{m}^2}{\text{sec}^2 \circ K}
\]

**Standard Atmosphere**

The international community has agreed upon a set of average atmospheric conditions known as the **standard atmosphere**. Conditions at sea level in the standard atmosphere are frequently referred to as being at **standard day conditions**. The standard atmosphere can be summarized through the following table for temperature.

<table>
<thead>
<tr>
<th>Elevation (km)</th>
<th>Temperature ©</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>0-11</td>
<td>15-6.5h</td>
</tr>
<tr>
<td>11-20.1</td>
<td>-56.5</td>
</tr>
<tr>
<td>20.1-32.2</td>
<td>-56.5+0.99(h-20.1)</td>
</tr>
<tr>
<td>32.2-47.3</td>
<td>-44.5+2.78(h-32.2)</td>
</tr>
<tr>
<td>47.3-52.4</td>
<td>-2.5</td>
</tr>
<tr>
<td>52.4-61.6</td>
<td>-2.5-1.96(h-52.4)</td>
</tr>
<tr>
<td>61.6-80.0</td>
<td>-20.5-3.91(h-61.6)</td>
</tr>
<tr>
<td>80-90</td>
<td>-92.5</td>
</tr>
</tbody>
</table>
EXAMPLE PROBLEM 3.2

Compute the pressure and density of the standard atmosphere from sea level to 10 kilometers.

Solution:

1. Solve the pressure-height differential equation (3.2b) using the perfect gas equation of state and the temperature variation equation.

\[
\frac{dp}{dz} = -\rho g \quad (a)
\]

\[
p = \rho RT \quad (b)
\]

\[
T = T_o + m(z-z_o) \quad (c)
\]

\[
\frac{dp}{z} = -\frac{p}{RT}g
\]

\[
\frac{dp}{p} = -\frac{gdz}{RT}
\]

\[
\frac{dp}{p} = -\frac{g}{R[T_o + m(z-z_o)]} = -\frac{g m \Delta(z-z_o)}{mR[T_o + m(z-z_o)]}
\]

\[
\ln\left(\frac{p}{p_o}\right) = -\frac{g}{mR} \ln\left[\frac{T_o + m(z-z_o)}{T_o}\right] = -\frac{g}{mR}\ln\left(\frac{T}{T_o}\right)
\]

\[
\frac{p}{p_o} = \left(\frac{T}{T_o}\right)^{-\frac{g}{mR}}
\]

Then apply the perfect gas equation of state to determine the density.
Example Problem 3.3

Figure 3.2a shows a sketch of a hydraulic jack. If a force $F_o = 100$ N is applied at a distance 8.0 cm from the pivot point "A", find the force $F_E$ at the bottom of the large piston.

Solution

1. Determine the force $F_1$ applied to the piston at point "B" by taking the moments about the point "A".

2. Determine the hydrostatic pressure at "B".

3. Determine the hydrostatic pressure at "C" as a function of $h_{BC}$.


5. Determine $p_E$ as a function of $p_D$ and $h_{DE}$.

Then given heights we can find $F_E$, or given $F_E$ we can find $h_{BE} = h_{BC} - h_{DE}$. 
To begin we recall from mechanics that if we neglect the weight of the bar and assume equilibrium we may write

\[ \Sigma M = 0 \]

\[ p = \frac{F}{A} \]

\[ p_B = -\rho g h_{CB} + p_C \]

\[ p_C = p_D \]

\[ p_E = -\rho g h_{DE} + p_D \]

\[ p_E = p_B + \rho g (h_{CB} - h_{DE}) \]

To begin we recall from mechanics that if we neglect the weight of the bar and assume equilibrium we may write

\[ \Sigma M_A = \]

\[ F_1 = \]

and if we then neglect the weight of the small piston we can find the force at "B" via

\[ F_B = F_1 = \]

\[ p_B = \]

\[ p_B = \]
If we then supply numbers; $h_{CB} = 10\, \text{cm}$, $h_{DE} = 6\, \text{cm}$ and $\gamma_{\text{OIL}} = 8400\, \text{N/m}^3$

$$P_E =$$

and

$$F_E = p_E A_E =$$

$$F_E =$$

c) **Manometric Measurement of Pressure**

A simple and effective way to measure pressure is to measure the height of the column of liquid supported by the pressure.

The accuracy of the measurement is related to the ability to measure small displacements.

**EXAMPLE PROBLEM 3.4**

Consider the manometer shown in the following figure. If $h_1 = h_4$, the device is referred to as a U-tube manometer.

Determine the pressure difference ($p_1 - p_2$) when

$\rho_1 = \rho_{\text{H}_2\text{O}}$ and $\rho_2 = \rho_{\text{Hg}}$

and both fluids are at $20\, ^\circ\text{C}$
Solution:

1. \( p_1 \) is related to \( p_A \) by the hydrostatic equation (use \( \rho_1 \)).
2. \( p_A \) is equal to \( p_B \) (same height in fluid 2).
3. \( p_B \) is related to \( p_3 \) by the hydrostatic equation (use \( \rho_2 \)).
4. \( p_3 \) is related to \( p_2 \) by the hydrostatic equation (use \( \rho_1 \)).

\[
\begin{align*}
    p_A &= -\rho_1 g(h_2 - h_1) + p_1 \\
    p_A &= p_B \\
    p_B &= -\rho_2 g(h_2 - h_3) + p_3 \\
    p_3 &= -\rho_1 g(h_3 - h_4) + p_2
\end{align*}
\]

We then use the above to solve for \( p_1 \)

\[
\begin{align*}
    p_1 &= p_A + \rho_1 g(h_2 - h_1) \\
    &= p_B + \rho_1 g(h_2 - h_1) \\
    &= p_3 - \rho_2 g(h_2 - h_3) + \rho_1 g(h_2 - h_1) \\
    &= -\rho_1 g(h_3 - h_4) - \rho_2 g(h_2 - h_3) + \rho_1 g(h_2 - h_1) + p_2 \\
    &= \rho_1 g(h_2 - h_1 - h_3 + h_4) + \rho_2 g(h_3 - h_2) + p_2
\end{align*}
\]

So that finally
Consider a U-tube manometer \((h_1 = h_4)\) and let \(h_3 = 2\text{m}\) and \(h_2 = 1\text{m}\)

\[ p_1 - p_2 = \]

QUESTION: If the accuracy of the pressure measurement depends on the accuracy of the displacement measurement, which system would you expect to be more accurate; One with large or small fluid density?

b) Forces on Structures and Submerged Bodies

- The hydrostatic equation allows one to calculate the force exerted on surfaces submerged in a fluid resulting from the hydrostatic pressure.
- Note, the pressure force is always acting perpendicular to the control surface.
EXAMPLE PROBLEM 3.5

What is the total hydrostatic force per unit width and the point of application of this force caused by the water acting on the dam shown at right on a STD? What is the moment generated by this force at points A, B & C?

Solution

1. Determine the resultant force acting on the dam by integrating the pressure distribution over the surface area.
2. Determine the point of action of this force by assuring that the moment generated by the resultant force exactly balances the integrated moments of the infinitesimal forces.
3. Calculate the moments at points A, B & C using the resultant forces and the geometry (just like in Mechanics).

We note that the pressure is the force per unit area, therefore, if the pressure changes infinitesimally we may write

\[ P = \frac{dF}{dA} = -\rho g y + P_{ref} \]

In terms of a free body diagram the local forces can then be visualized as shown at right.
Area calculation

\[ s = \sqrt{2}(y + 30m) \]
\[ A = s(1m) = \sqrt{2}(y + 30m)(1m) \]
\[ dA = \sqrt{2}dy(1m) \]

Resultant force

\[
F_R = \int_{A}^{C} p\,dA = \int_{A}^{0} (-\rho gy + p_{ref})\,dA
\]
\[ = \int_{-30m}^{0} (-\rho gy + p_{ref})\sqrt{2}dy(1m) \]
\[ = -\sqrt{2}\rho g \int_{-30m}^{0} y\,dy(1m) + \sqrt{2}p_{ref} \int_{-30m}^{0} dy(1m) \]
\[ = -\sqrt{2}\rho g \frac{1}{2}y^2(1m)_{-30m}^{0} + \sqrt{2}p_{ref} y(1m)_{-30m}^{0} \]

The point of action is such that the moment generated about any point is the same as the moment generated by the sum of infinitesimal forces i.e.

\[
M_R = \bar{s}F_R = \int_{A}^{0} \frac{dF}{dA} s\,dA
\]
\[ = \int_{-30m}^{0} (-\rho gy + p_{ref})\sqrt{2}(y + 30m)\sqrt{2}(1m)\,dy \]
\[ = 2m \left[ -\int_{-30m}^{0} \rho gy^2\,dy - \int_{-30m}^{0} \rho g(30m)y\,dy + \int_{-30m}^{0} p_{ref}y\,dy + \int_{-30m}^{0} p_{ref}(30m)\,dy \right] \]
Then

\[ F_R = \frac{\sqrt{2}}{2} \left( 1000 \frac{kg}{m^3} \right) (9.8 \frac{m}{sec^2}) (30m)^2 (1m) + \sqrt{2} (1.01 \cdot 10^5 Pa) (30m)(1m) \]

\[ F_R = \]

Next, \( M_A = M_R \)

\[ M_R = 2m \left[ -\frac{1}{3} \left( 1000 \frac{kg}{m^3} \right) (9.8 \frac{m}{sec^2}) y \right]_{-30m}^0 \]  
\[ + \frac{1}{2} \left( 1.01 \cdot 10^5 Pa \right) y \right]_{-30m}^0 \]  
\[ = 2m \left[ -\frac{1}{3} \left( 1000 \frac{kg}{m^3} \right) (9.8 \frac{m}{sec^2}) (30m)^3 + \frac{1}{2} \left( 1000 \frac{kg}{m^3} \right) (9.8 \frac{m}{sec^2}) (30m)(-30m)^2 \right. \]
\[ - \frac{1}{2} \left( 1.01 \cdot 10^5 Pa \right) (-30m)^2 - \left( 1.01 \cdot 10^5 Pa \right) (30m)(-30m) \]  
\[ M_R = \]

\[ \bar{S} = \frac{M_R}{F_R} \]

and we find \( M_C \) from

\[ M_C = -F_R(\sqrt{2} \cdot 30m - \bar{S}) = -1.052 \cdot 10^7 N (42.43m - 17.02m) \]

=
Finally, $M_B$ is found by determining the moment arm $\hat{s}$ using the diagram shown below, which leads to the equation

$$
\hat{s} = \sqrt{2}(15m) - \bar{s}
$$

So that we have

$$M_B = -(1.052 \cdot 10^7 N)(4.19m)$$
EXAMPLE PROBLEM 3.6

The end of a reservoir has the shape of a quarter circle of radius $R_o$. It is hinged at the bottom and restrained by a horizontal strap at the top.

Find the strap force, $T$, per unit width of reservoir.

Solution:

1. Determine the infinitesimal hydrostatic pressure along the shape as a function of $h$.
2. Determine the resultant force from these pressures (integrating as in the last example).
3. Determine the restraining force $T$ by assuming the system is in equilibrium so that you may sum the moments to zero (like Mechanics).

Now, the hydrostatic gauge pressure can be determined from

$$p = \rho \ g \ h$$
$$h = R_o \sin \theta$$
$$p = \rho \ g \ R_o \sin \theta$$
The restraining force $T$ applies a torque at point "O", which is equal and opposite to the torque generated by the hydrostatic pressure acting on the circular arc. So that if $TR_o$ is the moment generated at point "O" by the restraining force, we can form the equilibrium moment balance

$$TR_o = \text{Hydrostatic force} \times \text{effective distance (K)}$$

where $K = R_o \cos \theta$

$$TR_o = \int_A p(R_o d\theta) \cdot R_o \cos \theta$$

$$TR_o = \int_A pR_o^2 \cos \theta \ d\theta$$

substitute for the pressure

$$TR_o = \int_A \rho g R_o \sin \theta (R_o^2 \cos \theta) \ d\theta$$

$$TR_o = \int_0^{\pi/2} \rho g R_o^3 \sin \theta \cos \theta \ d\theta$$

$$TR_o = \rho g R_o^2 \left[ \frac{\sin^2 \theta}{2} \right]_0^{\pi/2} = \rho g R_o^2 \left[ \frac{1}{2} - 0 \right] = \frac{1}{2} \rho g R_o^2$$

$$T =$$

**QUESTION:** Does the exclusion of the reference pressure affect the result for $T$?