7.1 Introduction

The effects of fluid flow about some shape depend upon many variables. For example, if one considers the flow about a sphere, the force depends upon

1. 
2. 
3. 
4. 

etc.

Analyzing or testing all potential combinations of these parameters, even just enough to determine trends, would be enormously expensive. Dimensional analysis makes understanding the effects of multiple variable changes possible with just a minimal amount of analysis/testing. The idea is to search for or create a small number of independent variable combinations that when changed will give all of the information sought.

For example, when considering the drag force on a sphere, we might assume that it is related to the independent variables of importance by some functional relation

\[ F = \]

Dimensional analysis techniques will be used to show that a dimensionless force is related to a function of a single variable

\[ \frac{F}{\rho U^2 D^2} = f_1 \left( \frac{\rho UD}{\mu} \right) \]

In this way the changes in the dimensionless force can be determined through tests or analysis conducted for a few values of the single variable \( \frac{\rho UD}{\mu} \). The Buckingham Pi Theorem can be used to find combinations of this sort
7.2 Buckingham Pi Theorem

As stated earlier, we can assume that the drag force on a sphere can be related via an expression such as

\[ F = f(D, U, \rho, \mu) \]

Note that we have assumed this to be true, we must use some physical insight to make this assertion. Once it is established we can rewrite it in the more general form

\[ q_1 = \]

For more general problems we may write the dependent variable as a function of \( n-1 \) independent variables (\( n=5 \) for the sphere)

\[ q_1 = f(q_2, q_3, ..., q_n) \]

Which can be written equivalently as

\[ g(q_1, q_2, ..., q_n) = 0 \tag{7.1} \]

For the sphere we have \( g(\quad) = 0 \).

The Buckingham Pi Theorem states that (7.1) can be written in terms of a new function of \( n-m \) dimensionless products (the \( \Pi \)s) of the form

\[ G(\Pi_1, \Pi_2, ..., \Pi_{n-m}) = 0 \tag{7.2} \]

or equivalently

\[ \Pi_1 = G_1(\Pi_2, \Pi_3, ..., \Pi_{n-m}) \]
The value of $m$ is typically (although not always) the number of dimensions that make up the variables, i.e. Mass, Length, Time (MLt) or Force, Length, Time (FLt). The parameters are not unique and new ones can be made through various combinations. In addition, a parameter is not considered independent if it can be formed by some combination of the other parameters.

7.3 Implementing the Buckingham Pi Theorem

1. List all assumed variables.

2. Choose a primary dimension system (MLt, FLt).

3. List the variables in terms of the primary dimensions.

4. Choose a set of variables, equal in number to the primary dimensions, which include all of the primary dimensions.

5. Set up and solve for the dimensionless groups using powers of the variables in step 4 and each of the remaining variables in turn.

6. Recheck the dimensionality of the resulting groups.
EXAMPLE PROBLEM 7.1

Obtain the dimensionless groups for the drag force over a sphere.

Solution:

Apply the Buckingham Pi Theorem. (Note that this implementation is different from that chosen in the text.)

1. \( F \ U D \ \rho \ \mu \ n = \)

2. Choose \( MLt \) \( m = \)

3. \[ \begin{array}{ccc}
F & U & D \\
\rho & \mu & \\
\left[ \frac{L}{M} \right] & \left[ \frac{1}{M} \right] & \left[ \frac{1}{L^2} \right]
\end{array} \]

4. Choose \( D, \ \rho \) and \( \mu \)

5. The dimensionless parameters are found from

\[ \Pi_1 = D^a \rho^b \mu^c F = [L]^a \left[ \frac{M}{L^3} \right]^b \left[ \frac{M}{Lt} \right]^c \left[ \frac{ML}{t^2} \right] \]

To obtain a dimensionless variable we must have

\begin{align*}
L: \ a - 3b - c + 1 &= 0 \\
M: \ b + c + 1 &= 0 \quad \Rightarrow \quad c = , \ b = , \ a = \\
t: \ -c - 2 &= 0
\end{align*}
\[ \Pi_1 = D^0 \rho^1 \mu^{-2} F = \]

Similarly

\[ \Pi_2 = D^a \rho^b \mu^c U = [L]^a \left[ \frac{M}{L^3} \right]^b \left[ \frac{M}{Lt} \right]^c \]

\[
\text{L: } a - 3b - c + 1 = 0 \\
\text{M: } b + c = 0 \quad => \quad c = , b = , a = \\
\text{t: } -c - 1 = 0
\]

\[ \Pi_2 = \]

Therefore, since \( \Pi_1 = f(\Pi_2) \) we have

However, this is considerably different from the result in the text mentioned earlier. To match the text example we must simply divide the two parameters by \( \Pi_2^2 \).
7.4 Important Dimensionless Parameters

Reynolds Number \[ \text{Re} = \frac{\rho U L}{\mu} = \frac{UL}{v} \]

- ratio of inertial to viscous forces
- 

Mach Number \[ M = \frac{U}{c} \]

- ratio of velocity to speed of sound
- 

Euler Number \( C_p \) \[ E_{\mu} = \frac{\Delta p}{\frac{1}{2} \rho U^2} \]

- ratio of pressure forces to inertial forces
- 

\[ \Pi_1 = \frac{\Pi_1}{\Pi_2} = \frac{\frac{F \rho}{\mu^2}}{\frac{D^2 \rho^2 U^2}{\mu^2}} = \]

\[ \Pi_2 = \frac{\Pi_2}{\Pi_2^2} = \frac{\frac{D \rho U}{\mu}}{\frac{D^2 \rho^2 U^2}{\mu^2}} = \]
7.5 Flow Similarity

Dimensionless numbers are useful because they give us a way to compare flow fields with different conditions. In the case of flow past a sphere we had

\[
\frac{F}{\rho U^2 D^2} = f_1 \left( \frac{\rho UD}{\mu} \right) = f_1(\quad)
\]

A model in a wind tunnel can be used to determine the force on a geometrically similar large body, if we have

\[
Re_{\text{Large Body}} = Re_{\text{Small Prototype}}
\]

then

\[
\left( \frac{F}{\rho U^2 D^2} \right)_{LB} = \left( \frac{F}{\rho U^2 D^2} \right)_{SP}
\]

So that if

\[
D_{LB} = 100D_{SP}
\]

and the same fluid is used (i.e. \(\rho/\mu\) is the same), we must have

\[
\left( \frac{\rho UD}{\mu} \right)_{LB} = \left( \frac{\rho UD}{\mu} \right)_{SP}
\]

so that

\[
\frac{U_{LB}}{U_{SP}} = \quad
\]

and

\[
U_{SP} = \quad
\]

Hence, for Reynolds number equality we must have a fluid velocity than that of the actual body. Alternatively, we might choose to consider a different fluid. For example, if we require the force about a sphere in air at standard day

conditions we have $v_{atr} = 1.6 \cdot 10^{-5} m^2/sec$, $v_{water} = 1.0 \cdot 10^{-6} m^2/sec$, therefore if

$$\left( \frac{UD}{v} \right)_{LB} = \left( \frac{UD}{v} \right)_{SP}$$

then

$$\frac{U_{SP}}{U_{LB}} = \left( \frac{D_{LB}}{D_{SP}} \right) \left( \frac{v_{SP}}{v_{LB}} \right) =$$

So that now $U_{SP} =$.