8.1 Introduction

In earlier sections the flow in pipes and channels were assumed to be uniform, that is, the velocity was the same everywhere across the pipe. However, in chapter 1 we studied viscosity and the no-slip condition, which mandates that the velocity of the fluid at the wall be equal to that of the wall. Clearly these two ideas are contradictory. We will now consider "real" fluids, those with nonzero viscosity.

Flow at the Entrance to a Pipe

The flow field for a typical pipe is shown in the diagram below.

1. Flow at the entrance is uniform.
2. The no-slip condition causes the velocity near the pipe walls to slow, a separate boundary layer forms that is the region affected by the wall.
3. The boundary layers from the top and bottom walls grow until the entire flow field is viscous. The flow is then said to be
At any pipe cross section an average velocity can be defined as

\[
\bar{V} = \frac{\int_{A} u dA}{A} \quad (8.1)
\]

The continuity equation can be used to prove that

\[
\bar{V} = \;
\]

This value can then be used in empirical formulas to predict the distance at which fully developed flow occurs

\[
\frac{L}{D} \approx 0.06 \frac{\rho \bar{V} D}{\mu} \quad (8.2)
\]

We will analyze fully developed flows in this chapter and utilize some useful engineering expressions for their application. The first such flow was examined earlier in our introductory comments; fully developed laminar flow between infinite parallel plates.

8.2 Fully Developed Laminar Flow Between Infinite Parallel Plates With Arbitrary Pressure Gradient

In our earlier developments we derived the momentum equation for a control volume. If we apply the x direction equation to a fully developed pipe flow we have

\[
\Sigma F_x = \frac{\partial}{\partial t} \int_{\Omega} \rho u d\Omega = \int_{S} \rho u \bar{V} \cdot \hat{d}A
\]

1. For steady flows the first integral is zero.
2. Since the velocity profile is the same at each station, the momentum flux is constant. Therefore, the second integral is also zero.
We are left with an expression that is very similar to what we found in hydrostatics

\[ \Sigma F = 0 \]

However, in this case **shear forces are not zero**, since we have a flow. We must therefore consider the shear as well as the pressure forces in our analysis. If we consider the control volume shown at right, and assume that we know the pressure and shear at the center, we can form Taylor series expansions about the center of the volume.

![Diagram of forces acting on the fluid]

\[
F_p = \left( p - \frac{\partial p}{\partial x} \frac{dx}{2} + ... \right) dydz - \left( p + \frac{\partial p}{\partial x} \frac{dx}{2} + ... \right) dydz = -\frac{\partial p}{\partial x} dx dy dz + ...
\]

\[
F_s = \left( \tau_{yx} \frac{\partial \tau_{yx}}{\partial y} \right) dx dz - \left( \tau_{yx} \frac{\partial \tau_{yx}}{\partial y} \right) dx dz =
\]

Therefore

\[ \frac{\partial p}{\partial x} = \quad (8.3) \]

or

\[ \tau_{yx} = \]
and for a Newtonian fluid

\[ u = \left( \frac{\partial p}{\partial x} \right) y + C_1 \]

so that

\[ u = \quad (8.4) \]

Then using the boundary conditions

\[ u = \quad @ \quad y = 0 \quad \& \quad y = a \quad (\text{channel height}) \]

\[ = \frac{1}{2\mu} \left( \frac{\partial p}{\partial x} \right) + \frac{C_1}{\mu} + C_2 \]

\[ C_2 = \]

and

\[ 0 = \frac{1}{2\mu} \left( \frac{\partial p}{\partial x} \right) + \frac{C_1}{\mu} \]

\[ C_1 = \]
So that

\[ u = \frac{a^2}{2\mu} \left( \frac{\partial p}{\partial x} \right) \left[ \left( \frac{y}{a} \right)^2 - \left( \frac{y}{a} \right) \right] \]  

(8.5)

Velocity profile for variable pressure gradient
fully developed flow in a channel

Then

\[ \tau_{yx} = \]  

(8.6)

\[ Q = \]  

or

\[ Q = -\frac{l}{12\mu} \frac{\Delta p}{L} a^3 \]

\[ \bar{V} = \]  

(8.7)
Note that at $y = 0 \quad \tau_{yx} = -\frac{a}{2} \left( \frac{\partial p}{\partial x} \right)$. This initially appears to say that a net force in the "+" $x$ direction is exerted on the fluid. However, one must realize that $\frac{\partial p}{\partial x}$ for flow from left to right, therefore, a net retarding force is felt by the fluid. Hence, equation 8.6a determines a force on the fluid not on the wall. The forces on the pipe are found by noting that the pipe surface normal is in the opposite direction to the fluid normal, so that the shear force will also be in the opposite direction by virtue of the right hand rule.

Example 8.1

A 1 ft channel is to be used to transport water. At a point in the pipe far from the entrance a gauge reads a pressure of 120 psi. If the gauge could be moved 2000 ft downstream, what would it read if the volume flow rate is $300 \text{ ft}^3/\text{sec}$ per unit depth?

Solution:

1. Determine $\Delta p$ from equation 8.6c.

$$
\Delta p = \frac{-12\mu Q L}{1a^3}
$$
Hence

\[ P_{\text{down stream}} = \]

8.3 Fully Developed Laminar Flow Between Infinite Parallel Plates Moving at Constant Speed With Arbitrary Pressure Gradient

The previous analysis was quite general up until the boundary conditions were applied. Therefore, a moving wall can be incorporated easily if new boundary conditions are imposed.

\[ u = 0 \quad y = 0 \quad (a) \]
\[ u = U \quad y = a \quad (b) \]

The by virtue of (a), \( C_2 = 0 \) and from (b)

\[ U = \frac{1}{2\mu} \left( \frac{\partial p}{\partial x} \right) a^2 + \frac{C_1 a}{\mu} \]

So that

\[ u = \frac{1}{2\mu} \left( \frac{\partial p}{\partial x} \right) y^2 + \frac{Uy}{a} - \frac{1}{2\mu} \left( \frac{\partial p}{\partial x} \right) ay \]

\[ u = \frac{Uy}{a} + \frac{a^2}{2\mu} \left[ \left( \frac{y^2}{a} \right) - \left( \frac{y}{a} \right) \right] \]  \hspace{1cm} (8.7)
and similarly we have

\[
\tau_{xx} = \mu \frac{U}{a} = a \left( \frac{\partial p}{\partial x} \right) \left[ \frac{y}{a} - \frac{1}{2} \right]
\]

\[
Q = \frac{U l a}{2} - \frac{l}{12\mu} \left( \frac{\partial p}{\partial x} \right) a^2
\]

\[
\bar{V} = \frac{U}{2} - \frac{1}{12\mu} \left( \frac{\partial p}{\partial x} \right) a^2
\]

8.4 Fully Developed Laminar Flow in a Pipe

Consider now a circular pipe. The control volume is then

Once again Newton’s 2nd law yields

\[
\sum F = 0
\]

and

\[
F_p = \left( p - \frac{\partial p}{\partial x} \frac{dx}{2} \right) 2\pi r dr - \left( p + \frac{\partial p}{\partial x} \frac{dx}{2} \right) 2\pi r dr
\]

\[
F_s = \left( \tau_{rx} + \frac{d\tau_{rx}}{dr} \frac{dr}{2} \right) 2\pi \left( r + \frac{dr}{2} \right) dx - \left( \tau_{rx} - \frac{d\tau_{rx}}{dr} \frac{dr}{2} \right) 2\pi \left( r - \frac{dr}{2} \right) dx
\]

or

So that
\[
\frac{\partial p}{\partial x} = \frac{\tau_{nx}}{r} + \frac{d\tau_{nx}}{dr} = \frac{1}{r} \frac{d(r\tau_{nx})}{dr}
\]  \quad (8.9)

Again integration yields the velocity profile from the boundary conditions

\[ u = 0 \quad r = R \]

and

\[ u \text{ finite} \quad r = 0 \]

Finally

\[ u = -\frac{R^2}{4\mu} \left( \frac{\partial p}{\partial x} \right) \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \]  \quad (8.10)

Now in a manner similar to that of the last example we can write

\[ \tau_{nx} = \frac{r}{2} \left( \frac{\partial p}{\partial x} \right) \]  \quad (8.11)

\[ Q = -\frac{\pi R^4}{8\mu} \left( \frac{\partial p}{\partial x} \right) \]

\[ \bar{\nu} = -\frac{R^2}{8\mu} \left( \frac{\partial p}{\partial x} \right) \]
8.5 Laminar and Turbulent Flows

Experimental evidence has shown that there are two general classes of flow

1. - Flow travels smoothly in layers.
2. - Flow mixes between layers in a random fashion.

The transition between the two classes of flow typically depends on a nondimensional parameter known as the **Reynolds number**, \( R_e \). For pipe flow it is defined as

\[
R_e = \frac{\rho \bar{V} D}{\mu} \tag{8.12}
\]

- \( \rho \) - density
- \( \bar{V} \) - average velocity
- \( D \) - pipe diameter
- \( \mu \) - viscosity

Typically we can expect laminar flow for , although this number can change due to such effects as surface roughness.

Among other changes, the velocity profile changes drastically for turbulent flows, hence \( \tau_w \) also changes. This necessitates the changes in the empirical relations described below.

8.6 Head Loss

Earlier in the section on the extended Bernoulli equation, head loss due to friction was shown to vary proportionally to \( V^2 \). Head loss is typically written in the following form

\[
h_l = f \frac{L \bar{V}^2}{D \frac{1}{2}} \tag{8.13}
\]
Note that in this case head loss has the units of energy/mass not energy/weight as was commonly used earlier. **BEWARE!**

The value of $f$ has been tabulated empirically and can be determined via table look-up, such as can be performed with figure 8.14 on page 350 of the text.

**Laminar Flow**

The value of $f$ for laminar flow is given by

$$ f_{laminar} = \quad (8.14) $$

This can be easily found by using the result for pipe flow equation 8.11

$$ \Delta p = \frac{128\mu LQ}{\pi D^4} $$

$$ = \frac{128\mu L\bar{V}}{\pi D^4} \left(\frac{\pi D^2}{4}\right) $$

$$ \Delta p = 32 \frac{L \mu \bar{V}}{D^4} $$
Or in terms of energy/mass head loss as

\[ h_i = 32 \frac{L \mu \bar{V}}{D \rho D} = \frac{L}{D} \left( \frac{\bar{V}^2}{2} \right) \left( \frac{64 \mu}{\rho \bar{V} D} \right) \]

\[ h_i = \left( \frac{L}{D} \right) \left( \frac{\bar{V}^2}{2} \right) \left( \frac{64 \mu}{\rho \bar{V} D} \right) \]  \hspace{1cm} (8.14)

**Turbulent Flow**

Dimensional analysis can be used to show that

\[ f = \Phi( \quad ) \]

where \( \frac{e}{D} \) is the relative roughness of the pipe. This value can be determined for various construction products using figure 8.15 on page 351 of the text. The charts have been combined into the convenient equation

\[ \frac{1}{\sqrt{f}} = -2 \log_{10} \left( \frac{e}{D} \right) + \frac{2.51}{R_e \sqrt{f}} \]  \hspace{1cm} (8.15)

which must be solved numerically since it is transcendental. See the text for more details of this solution approach.
EXAMPLE 8.2

Determine the head loss in a 10 in diameter concrete water pipe with $e = 0.01\,ft$ over a pipe length of $100\,ft$ if the volume flow rate is $0.5\,ft^3/sec$

Solution:

1. Compute $\bar{V}$ from $Q$.
2. Compute $Re$ to determine if the flow is laminar or turbulent.
3. Compute $e/D$.
4. If the flow is laminar determine $f$ from equation 8.14, if turbulent consult the chart on pg 350.
5. Apply equation 8.13 to determine $h_l$

\[ Q = \bar{V}A \quad \rightarrow \quad \bar{V} = \frac{Q}{A} \]

\[ \bar{V} = \]

\[ Re = \frac{\rho \bar{V}D}{\mu} \]

\[ = \]

\[ Re = \]

Therefore the flow is
Then to use the tables we must compute

\[
\frac{e}{D} = \frac{0.01 \text{ft} \cdot 12\text{in}}{10\text{in} \cdot 1\text{ft}}
\]

\[
\frac{e}{D} =
\]

\[
h_i =
\]

\[
=
\]

Note that the total energy/weight head required to overcome this head loss is