Conservation of Mass
Consider a fixed control volume and an infinitesimal surface element.

Hence the mass entering the volume is

\[
\frac{\partial}{\partial t} \int_V \rho dV
\]

(2.3)

Continuity Equation
Newton’s Second Law

Apply Newton’s second law to a fixed mass of fluid. If $m$ is a constant we get

$$\vec{F} =$$ (2.5)

Consider now the volume surrounding the fixed mass of fluid.

The forces that act on this volume are

- Forces which are proportional to the mass of fluid.
- Forces which act only on the boundary of the volume.

$$\vec{F} =$$

(2.6)
Consider next the acceleration term in (2.5). Note that velocity is a function of position and time, hence

\[ d\vec{V} = \]  

(2.7)

(2.8)

By definition \( u = v = w = \)

Substitute (2.8) into (2.7) \( d\vec{V} = \)

Divide by dt \( \frac{d\vec{V}}{dt} = \)  

(2.9)

If we use (2.9) in (2.5), integrate of the entire volume and represent the forces using equation (2.6), we get

\[ \iiint \frac{\partial (\rho \vec{V})}{\partial t} dV + \int (\rho \vec{V} \cdot d\vec{S}) \vec{V} = \iiint p dV - \iiint p dS \]  

(2.10)
Conservation of Energy

The conservation of energy is essentially the first law of thermodynamics applied to a fixed control volume

\[ \int_V \dot{q} \rho dV = \int_S p \vec{V} \cdot \hat{n} ds \]

\[ = \int_V \rho \vec{f} \cdot \vec{V} dV \]

\[ = \int_S \left( \rho \vec{V} \cdot \hat{n} ds \right) \left( e + \frac{V^2}{2} \right) \]

Hence,

\[ \int_V \dot{q} \rho dV = \int_S p \vec{V} \cdot \hat{n} ds + \int_V \rho \vec{f} \cdot \vec{V} dV \]

\[ = \int_V \partial \left( \rho \left( e + \frac{V^2}{2} \right) \right) dV + \int_S \left( \rho \vec{V} \cdot \hat{n} ds \right) \left( e + \frac{V^2}{2} \right) \]

(2.11)
Differential Forms

Equations (2.4), (2.10) and (2.11) are the integral forms of the governing equations. They apply to finite volumes of fluid. Differential equations can be formed from them that are applicable at points in the field. These forms are discussed in detail in chapter 6. They require the use of the equations

\[ \iiint_V \nabla \cdot \vec{A} \, dV = \iint_S \vec{A} \cdot d\vec{S} \quad (6.1) \]

and

\[ \iiint_V \vec{\nabla} \phi \, dV = \iint_S \phi \, d\vec{S} \quad (6.2) \]

Continuity Equation

\[ \frac{\partial}{\partial t} \iiint_V \rho \, dV + \iint_S (\rho \vec{v} \cdot d\vec{S}) = 0 \]

Apply (6.1)

\[ \iiint_V \left[ \frac{\partial}{\partial t} \rho + \rho \vec{v} \cdot \nabla \right] dV = 0 \]

If the above equation is true it must apply to any volume. We can then say

\[ (6.3) \]

Momentum Equation

\[ \iiint_V \frac{\partial (\rho \vec{v})}{\partial t} \, dV + \iint_S (\rho \vec{v} \cdot d\vec{S}) \vec{v} = \iint_V \rho \vec{g} \, dV + \iint_S \rho \vec{f} \cdot d\vec{S} \]

Apply (6.2) to

Surface Integrals

\[ \iiint_V \left[ \frac{\partial}{\partial t} \rho \vec{v} + \rho \vec{v} \cdot \nabla \vec{v} \right] dV = 0 \]
Likewise,

\[ \frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V}) = \rho f_x - \frac{\partial p}{\partial x} \] \hspace{1cm} (6.4)

Likewise,

\[ \frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v \vec{V}) = \rho f_y - \frac{\partial p}{\partial y} \] \hspace{1cm} (6.5)

\[ \frac{\partial (\rho w)}{\partial t} + \nabla \cdot (\rho w \vec{V}) = \rho f_z - \frac{\partial p}{\partial z} \] \hspace{1cm} (6.6)

Finally, a similar use of equations (6.1) and (6.2) gives

\[ \frac{\partial}{\partial t}\left[ \rho \left( e + \frac{V^2}{2} \right) \right] + \nabla \cdot \left[ \rho \vec{V} \left( e + \frac{V^2}{2} \right) \right] = - \nabla \cdot (p \vec{V}) + \rho \dot{q} + \rho f \cdot \vec{V} \] \hspace{1cm} (6.7)

Equations (6.3), (6.4), (6.5), (6.6) and (6.7) are in since they all contain divergences of the fluxes \( \rho \vec{V}, \rho u \vec{V}, \rho v \vec{V}, \rho w \vec{V}, \rho \vec{V} (e + V^2/2) \). This form of the equations is also known as the .

A second differential form of the governing equations can be obtained by using the definition of the substantial derivative together with the identity

\[ \nabla \cdot (a \vec{B}) = a \nabla \cdot \vec{B} + \vec{B} \cdot \nabla a \] \hspace{1cm} (6.8)

Equation (6.3) using (6.8)

Nonconservative form of continuity equation

\[ \rho \frac{Du}{Dt} = - \frac{\partial p}{\partial x} + \rho f_x \] \hspace{1cm} (6.9)
\[ \rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y \quad (6.11) \]
\[ \rho \frac{ Dw }{ Dt } = -\frac{\partial p}{\partial z} + \rho f_z \quad (6.12) \]

Nonconservative forms of the

\[ \rho \frac{D}{Dt} \left( e + \frac{V^2}{2} \right) = -\nabla \cdot (p \vec{V}) + \rho \dot{q} + \rho (\vec{f} \cdot \vec{V}) \quad (6.13) \]

\[ \rho \frac{De}{Dt} = -p \nabla \cdot \vec{V} + \rho \dot{q} \]

\[ \rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \rho \dot{q} \]

\[ \rho \frac{Dh_o}{Dt} = \frac{\partial p}{\partial t} + \rho \dot{q} + \rho (\vec{f} \cdot \vec{V}) \]
Crocco’s Theorem

Crocco’s theorem asserts that. These factors are related for inviscid flows with in body forces through the equation

\[ T \nabla s = \nabla h_o - \vec{V} x (\nabla x \vec{V}) + \frac{\partial \vec{V}}{\partial t} \quad (6.14) \]

Proof:

\[ \rho \frac{D\vec{V}}{Dt} = - \nabla p \]

\[ T \nabla s = \nabla h - \frac{\nabla p}{\rho} \]

\[ T \nabla s = \nabla h + \frac{1}{\rho} \left[ \rho \frac{D\vec{V}}{Dt} \right] \]

\[ = \nabla h + \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \]

\[ h_o = h + \frac{V^2}{2} \]

\[ T \nabla s = \nabla h_o - \nabla \left( \frac{V^2}{2} \right) + \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \]

\[ T \nabla s = \nabla h_o - \vec{V} x (\nabla x \vec{V}) + \frac{\partial \vec{V}}{\partial t} \]