Text Chapter 9

Perturbation Potential Equation

Velocity Potential Equation

\[
\left(1 - \frac{\Phi_x^2}{a^2}\right) \Phi_{xx} + \left(1 - \frac{\Phi_y^2}{a^2}\right) \Phi_{yy} + \left(1 - \frac{\Phi_z^2}{a^2}\right) \Phi_{zz} \]
\[
- 2 \frac{\Phi_x \Phi_y}{a^2} \Phi_{xy} - 2 \frac{\Phi_x \Phi_z}{a^2} \Phi_{xz} - 2 \frac{\Phi_y \Phi_z}{a^2} \Phi_{yz} = 0
\]

Perturbation Velocity Potential Equation

\[
\left(1 - \frac{(V_\infty + \Phi_x)^2}{a^2}\right) \Phi_{xx} + \left(1 - \frac{\Phi_y^2}{a^2}\right) \Phi_{yy} + \left(1 - \frac{\Phi_z^2}{a^2}\right) \Phi_{zz} \]
\[
- 2 \frac{V_\infty + \Phi_x}{a^2} \Phi_y \Phi_{xy} - 2 \frac{V_\infty + \Phi_x}{a^2} \Phi_z \Phi_{xz} - 2 \frac{\Phi_y \Phi_z}{a^2} \Phi_{yz} = 0
\]

Linearized Perturbation Potential Equation

\[
\beta^2 \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0
\] (9.12)

\[
\beta^2 = 1 - M_\infty^2
\]

Linearized Pressure Coefficient

\[
C_p = -\frac{2\Phi_x}{V_\infty}
\] (9.20)
To derive the perturbation potential equation.

\[ u = V_\infty + u', \quad v = v', \quad w = w' \quad (9.1) \]

\[ \nabla \Phi = (V_\infty + u')\hat{i} + v\hat{j} + w\hat{k} \quad (9.2) \]

\[ u' = \phi_x', \quad v' = \phi_y', \quad w' = \phi_z' \quad (9.3) \]

\[ \Phi = V_\infty x + \phi \quad (9.4) \]

\[ \Phi_x = V_\infty + \phi_x \]

\[ \Phi_y = \phi_y \]

\[ \Phi_z = \phi_z \quad (9.5) \]

\[ \Phi_{xx} = \phi_{xx}, \quad \Phi_{yy} = \phi_{yy}, \quad \Phi_{zz} = \phi_{zz} \quad (9.6) \]

Substitute the above into the velocity potential equation to get the perturbation potential equation.

\[ \frac{h_\infty}{2} + \frac{V_\infty^2}{2} = h + \frac{V^2}{2} \quad (9.8) \]

\[ \frac{a_\infty^2}{\gamma - 1} + \frac{V_\infty^2}{2} = \frac{a^2}{\gamma - 1} + \frac{1}{2}(V_\infty + \phi_x)^2 + \phi_y^2 + \phi_z^2 \quad (9.9) \]

\[ a^2 = a_\infty^2 - \frac{(\gamma - 1)}{2} (2V_\infty \phi_x + \phi_x^2 + \phi_y^2 + \phi_z^2) \quad (9.10) \]
The goal is to eliminate all terms except those in the very first line above. This can be done because

Terms Small if

1. Underlined terms

2. $M_\infty^2 (\gamma + 1) \frac{u'}{V_\infty} u'_{x}$

3. $M_\infty^2 (\gamma - 1) \frac{u'}{V_\infty} v'_{y}$

4. $M_\infty^2 \left[ \frac{v'}{V_\infty} \left( \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right) + \frac{w'}{V_\infty} \left( \frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} \right) \right]$
Therefore, by excluding and flow equation (9.12) is obtained. To get the pressure coefficient note that

**Definition**

\[ C_p = \frac{2}{\gamma M_a^2} \left( \frac{p}{p_o} - 1 \right) \]  \hspace{1cm} (9.13)

\[ \frac{p}{p_o} = \left( \frac{T}{T_o} \right)^{\frac{\gamma}{\gamma - 1}} \]  \hspace{1cm} (9.14)

\[ \frac{T}{T_o} = 1 + \frac{\gamma - 1}{2} \frac{V_o^2 - V^2}{a_o^2} \]  \hspace{1cm} (9.15)

\[ \frac{T}{T_o} = 1 - \frac{\gamma - 1}{2 a_o^2} \left( 2 u' V_o + u'^2 + v'^2 + w'^2 \right) \]  \hspace{1cm} (9.16)

\[ \frac{p}{p_o} = \left[ 1 - \frac{\gamma - 1}{2} M_a^2 \left( \frac{2 u'}{V_o} + \frac{u'^2 + v'^2 + w'^2}{V_o^2} \right) \right]^{\frac{\gamma}{\gamma - 1}} \]  \hspace{1cm} (9.17)

\[ \frac{p}{p_o} = \left[ 1 - (\gamma - 1) M_a^2 \frac{u'}{V_o} \right]^{\frac{\gamma}{\gamma - 1}} \]  \hspace{1cm} (9.18)

\[ \frac{p}{p_o} = \gamma M_a^2 \frac{u'}{V_o} + \ldots \]  \hspace{1cm} (9.19)
Subsonic Similarity Rules

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Affinely Related Bodies

Two bodies are affinely related if they are represented by \((x,y)\) and \((x',y')\) such that constants \(\lambda_x\) and \(\lambda_y\) provide the relations

\[
x' = \lambda_x x, \quad y' = \lambda_y y
\]

The goal of these exercises is to relate a compressible airfoil to an identical incompressible airfoil. In that way testing can be conducted on the incompressible case at a lower cost and extrapolated to the compressible case.
Define a surface by \( y = f(x) \). Then
\[
\begin{align*}
\frac{df}{dx} &= \frac{\nu}{u} = \frac{\nu}{V_\infty} \quad \text{(9.20)} \\
\frac{df}{dx} &= \frac{\nu'}{V_\infty + u'} = \frac{\nu'}{1 + \frac{u'}{V_\infty}} \quad \text{(9.21)}
\end{align*}
\]

The tangency boundary condition is:
\[
\frac{df}{dx} = \frac{\nu'}{V_\infty} \quad \text{or} \quad \phi_y = V_\infty \frac{df}{dx}
\]

Now consider the two-dimensional linearized velocity potential equation:
\[
\beta^2 \phi_{xx} + \phi_{yy} = 0
\]

Introduce Transformation:
\[
\xi = x \quad \eta = \beta y
\]

Define
\[
\phi(\xi, \eta) = B \phi(x, y)
\]

Define
\[
\phi_x = \frac{\partial \phi}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \phi}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial \xi} = \frac{1}{B} \phi_\xi \quad \text{(9.25)}
\]
\[
\phi_{xx} = \frac{1}{B} \phi_{\xi \xi} \quad \text{(9.26)}
\]
\[
\phi_y = \beta \frac{\partial \phi}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \phi}{\partial \eta} \frac{\partial \eta}{\partial y} = \beta \phi_\eta = \frac{\beta}{B} \phi_\eta \quad \text{(9.27)}
\]
\[
\phi_{yy} = \frac{\beta^2}{B} \phi_{\eta \eta} \quad \text{(9.28)}
\]

The linearized velocity potential equation then becomes
\[
\phi_{\xi \xi} + \phi_{\eta \eta} = 0 \quad \text{(9.29)}
\]
Next only boundary conditions for the comparable incompressible airfoil are needed.

\[ \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{d\xi} \left( \frac{\eta}{\beta} \right) = \frac{1}{\beta} \frac{d\eta}{d\xi} \]  
\[ (9.30) \]

\[ \frac{df}{dx} = \frac{1}{\beta} \frac{dq}{d\xi} \]  
\[ (9.31) \]

\[ \phi_y = -\nu \frac{df}{dx} \]  
\[ (9.32) \]

\[ \frac{B}{B} \phi_\eta = \nu \frac{1}{\beta} \frac{dq}{d\xi} \]  
\[ (9.33) \]

\[ \bar{\phi}_\eta = \nu \frac{dq}{d\xi} \frac{B}{\beta^2} \]  
\[ (9.34) \]

Therefore, if we set \( B = \beta^2 \) \& \( \nu = \bar{\nu} \) we get

\[ \bar{\phi}_\eta = \bar{\nu} \frac{dq}{d\xi} \]  
\[ (9.35) \]

\[ q(\xi) = \eta = \beta y = \beta f(x) \]

\[ f(x) = \frac{1}{\beta} q(\xi) \]  
\[ (9.36) \]

Therefore, the compressible and incompressible airfoil shapes are

\[ \frac{t_l}{t_c} = \beta \]
Gothert's rule is found by noting

\[
\frac{C_{PC}}{C_{PI}} = \frac{-\frac{2}{V_\infty} \Phi_x}{\Phi_\xi} = \frac{-\frac{2}{V_\infty} \Phi_x}{B \Phi_\xi} = \frac{1}{B} = \frac{1}{\beta^2}
\]  \( \text{(9.37)} \)

\[
C_{PC} = \frac{C_{PI}}{(1 - M_\infty^2)}
\]  \( \text{(9.38)} \)

\[
C_L = \frac{L}{\frac{1}{2} \rho_\infty V_\infty^2 l} = \bar{C}_p \bar{d} \left( \frac{x}{L} \right)
\]  \( \text{(9.39)} \)

\[
\frac{C_{LC}}{C_{LI}} = \frac{1}{1 - M_\infty^2}
\]  \( \text{(9.40)} \)

\[
\frac{C_{MC}}{C_{MI}} = \frac{1}{1 - M_\infty^2}
\]  \( \text{(9.41)} \)

However, we want to relate our airfoil to the same incompressible airfoil.

\[
\frac{\gamma_{I_1}}{\gamma_{I_2}} = \frac{\alpha_{I_1}}{\alpha_{I_2}} = \frac{\theta_{I_1}}{\theta_{I_2}} = \beta
\]  \( \text{(9.42)} \)

\[
\frac{C_{PI_1}}{C_{PI_2}} = \frac{C_{LI_1}}{C_{LI_2}} = \frac{C_{MI_1}}{C_{MI_2}} = \beta
\]  \( \text{(9.43)} \)
\[
\frac{C_{PC}}{C_{PI_2}} = \frac{C_{PC}}{C_{PI_1}} \frac{C_{PI_1}}{C_{PI_2}} = \frac{1}{\beta^2} = \frac{1}{\beta}
\] (9.44)

\[
\frac{C_{PC}}{C_{PI}} = \frac{C_{Le}}{C_{LI}} = \frac{C_{Mc}}{C_{MI}} = \frac{1}{\sqrt{1 - M_\infty^2}}
\] (9.45)
Improved Corrections

\[ C_p = \frac{C_{p,o}}{\sqrt{1 - M_\infty^2} + \frac{M_\infty^2}{2\sqrt{1 - M_\infty^2}} C_{p,o}} \]

\( C_p \) is altered in a manner similar to the Prandtl-Glauret rule but local values of Mach number are used. (See attached Reader's Report).

\[ C_p = \frac{C_{p,o}}{\sqrt{1 - M_\infty^2} + \frac{M_\infty^2}{1 + \sqrt{1 - M_\infty^2}} C_{p,o}} \]

This idea makes use of the approximation. i.e. \( i \) is replaced by Laitone's correction

is preferred. In addition, the tangent gas approximation results in an effective maximum velocity, since negative pressures are possible (via \( C_p \) relation.)
Wavy Wall Problem

The result is

\[
C_{p\text{wall}} = \frac{2c}{\sqrt{1 - M_\infty^2}} \left( \frac{2\pi}{l} \right) \sin \left( \frac{2\pi x}{l} \right)
\]

Subsonic

Which implies

Equation (9.46), therefore, we might attempt to separate variables to find its solution.

\[
(1 - M_\infty^2) \phi_{xx} + \phi_{yy} = 0 \quad (9.46)
\]

\[
v(x,0) = (\phi_y)_{y=0} \approx V_\infty \left( \frac{dy}{dx} \right)_{\text{wall}} = V_\infty \varepsilon \sin \alpha x
\]

\[
\phi_x, \phi_y
\]

Equation (9.46), therefore, we might attempt to separate variables to find its solution.

\[
\phi(x,y) = F(x)G(y) \quad (9.48)
\]

\[
m^2 F''/G + FG'' = 0 \quad (9.49)
\]

\[
\frac{F''}{F} + \frac{G''}{m^2 G} = 0 \quad (9.50)
\]

The F terms are functions only of x and the G only of y. But (9.50) must hold for all a and y. Therefore the only solution is

\[
\frac{F''}{F} = -k, \quad \frac{G''}{m^2 G} = +k^2 \quad (9.51)
\]
\[ F = A_1 \sin kx + A_2 \cos kx \quad (9.52) \]

\[ G = B_1 e^{-mkty} + B_2 e^{mkty} \quad (9.53) \]

Which leads to

\[ B_2 = 0 \quad \text{Finite } \phi_x, \phi_y \text{ at } \infty \]
\[ A_1 = 0 \quad \text{Boundary Condition} \]
\[ k = \alpha \]
\[ A_2 = -\frac{V_\infty \varepsilon \alpha}{B_1 mk} \quad v'_{\text{wall}} = F(x) G'(y = 0) \]

\[ \phi(x,y) = FG = -\frac{V_\infty \varepsilon \alpha}{B_1 mk} \cos kxB_1 e^{-mkty} \quad (9.54) \]

\[ \phi(x,y) = -\frac{V_\infty \varepsilon}{\sqrt{1 - M_\infty^2}} \cos(\alpha x)e^{-\alpha \sqrt{1 - M_\infty^2}} \quad (9.55) \]

\[ u'(x,y) = \frac{V_\infty \varepsilon \alpha}{\sqrt{1 - M_\infty^2}} \sin(\alpha x)e^{-\alpha \sqrt{1 - M_\infty^2}} \quad (9.56) \]

\[ v'(x,y) = V_\infty \varepsilon \alpha \cos(\alpha x)e^{-\alpha \sqrt{1 - M_\infty^2}} \quad (9.57) \]

\[ C_p(x,y) = -\frac{2\varepsilon \alpha}{\sqrt{1 - M_\infty^2}} \sin(\alpha x)e^{-\alpha \sqrt{1 - M_\infty^2}} \quad (9.58) \]

Evaluation of equation (9.58) at \( y=0 \) yields the wall \( C_p \).
Linearized Supersonic Flow

\[(1 - M_\infty^2)\phi_{xx} + \phi_{yy} = 0\]

\[\lambda^2 \phi_{xx} - \phi_{yy} = 0 \quad \text{where} \quad \lambda^2 = \sqrt{M_\infty^2 - 1}\]

The governing equations for supersonic flows are hyperbolic. Solutions to the wave type equations were shown earlier to have the form

\[\phi(x,y) = f(x - \lambda y) + g(x + \lambda y)\quad \text{(9.61)}\]

Which implies that \(f = C_1\) along \(x - \lambda y = C_2\) lines and \(g = C_3\) along \(x + \lambda y = C_4\) lines. \(f\) and \(g\) are arbitrary functions that are defined by the boundary conditions. Note that these constant \(f\) and \(g\) lines are very similar to the nonlinear characteristic lines but their slopes have a constant value.

Note that the derivatives of \(\phi\) give the perturbation velocities, therefore, in the freestream \(\phi = C\). In addition, the contributions of \(f\) and \(g\) to \(\phi\) are felt only along the lines shown in the figure.
For flow above the surface

$$\phi(x,y) = f(x - \lambda y) + C$$  \hspace{1cm} (9.62)

Lines of constant \( \phi \) are then given by \( x - \lambda y = C \) so that

$$\frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}}$$  \hspace{1cm} (9.63)

Similar results hold for flow below a body. Next, consider again flow above a body.

$$u' = \phi_x = \frac{df}{d(x - \lambda y)} \frac{\partial(x - \lambda y)}{\partial x} = f'$$  \hspace{1cm} (9.64)

$$v' = \phi_y = \frac{df}{d(x - \lambda y)} \frac{\partial(x - \lambda y)}{\partial y} = -\lambda f'$$  \hspace{1cm} (9.65)

$$u' = -\frac{v'}{\lambda}$$  \hspace{1cm} (9.66)

$$\tan \theta = \frac{dy}{dx} = \frac{v'}{V_\infty + u'}$$  \hspace{1cm} (9.67)

$$v' = V_\infty \theta$$  \hspace{1cm} (9.68)

$$u' = -\frac{V_\infty \theta}{\lambda}$$  \hspace{1cm} (9.69)

$$C_p = \frac{-2\theta}{\sqrt{M_\infty^2 - 1}}$$  \hspace{1cm} (9.70)

Supersonic Linearized Small Perturbation Pressure Coefficient

$$C_p = \frac{-2\theta}{\sqrt{M_\infty^2 - 1}}$$  \hspace{1cm} (9.71)
So that for a circular arc airfoil
Supersonic Wavy Wall

Result

\[ C_{p_{wall}} = \frac{2 \varepsilon \alpha}{\sqrt{M_\infty^2 - 1}} \cos(\alpha x) \]  

(9.71)

\[ \phi_y = V_\infty \frac{dy}{dx} \]  

(9.72)

\[ - \lambda f'(x - \lambda y)_{y=0} = - \lambda f'(x) = V_\infty \varepsilon \alpha \cos(\alpha x) \]  

(9.73)

\[ f'(x) = - \frac{V_\infty \varepsilon \alpha \cos(\alpha x)}{\sqrt{M_\infty^2 - 1}} \]  

(9.74)

\[ f(x) = - \frac{V_\infty \varepsilon}{\sqrt{M_\infty^2 - 1}} \sin(\alpha x) + C \]  

(9.75)

\[ \phi(x,y) = f(x - \lambda y) = - \frac{V_\infty \varepsilon}{\sqrt{M_\infty^2 - 1}} \sin(\alpha[x - \lambda y]) + C \]  

(9.76)

\[ u'(x,y) = - \frac{V_\infty \varepsilon \alpha}{\sqrt{M_\infty^2 - 1}} \cos(\alpha[x - \sqrt{M_\infty^2 - 1}y]) \]  

\[ v'(x,y) = - V_\infty \varepsilon \alpha \cos(\alpha[x - \sqrt{M_\infty^2 - 1}y]) \]  

\[ C_{p(x,y)} = \frac{2 \varepsilon \alpha}{\sqrt{M_\infty^2 - 1}} \cos(\alpha[x - \sqrt{M_\infty^2 - 1}y]) \]

Note that the perturbations extend to infinity and are constant along the freestream Mach lines.